

# Quaternionic Quantum Interferometry

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**Abstract.** If scattering amplitudes are ordinary complex numbers (not quaternions) there is a universal algebraic relationship between the six coherent cross sections of any three scatterers (taken singly and pairwise). A violation of this relationship would indicate either that scattering amplitudes are quaternions, or that the superposition principle fails. Some possible experimental tests involve neutron interferometry,  $K_S$ -meson regeneration, and low energy proton-proton scattering.

When we progress in the hierarchy of numbers, we encounter integers, real numbers, complex numbers, and then quaternions. The latter are hypercomplex numbers which can be written as  $a + ib + jc + kd$ , where  $i^2 = j^2 = k^2 = -1$  and  $ij = -ji = k$ , etc. They are the only generalization of complex numbers that satisfies the associative and distributive laws, and for which division is possible and unique (Chevalley, 1946). They were originally introduced in classical physics by Hamilton, in order to describe 3-dimensional rotations.

When we further progress from classical physics to quantum theory, we learn that the states of a physical system can be represented by a linear manifold (Peres, 1993). Namely, if  $\psi_1$  and  $\psi_2$  are two possible states of a quantum system, and  $c_1$  and  $c_2$  are arbitrary numbers, then the expression  $c_1\psi_1 + c_2\psi_2$  also represents a possible state of that system. It is usually taken for granted that the coefficients  $c_1$  and  $c_2$  are complex numbers. However, it is possible to imagine a real quantum theory (Stueckelberg, 1960) or one based on quaternions (Finkelstein, Jauch, Schiminovich and Speiser, 1962–3; Emch, 1963; Wolff, 1981; Sharma and Coulson, 1987). The purpose of this article is to show how interferometric experiments can distinguish between these various quantum theories.

Real quantum theory, although logically consistent, can be easily ruled out for our world: e.g., complex coefficients are needed in order to combine linearly polarized photons into circularly polarized ones. More generally, correspondence with classical physics leads to the commutation relation  $[q, p] = i\hbar$ . [Here, it may be pointed out that Stueckelberg’s “real” quantum theory requires the introduction

of an operator  $J$  satisfying  $J^2 = -\mathbb{1}$  and commuting with all observables. As a consequence, the states  $\psi$  and  $J\psi$  are linearly independent, although they are physically indistinguishable. Moreover, the definition of the scalar product involves  $i$  explicitly: see his equation (A-2.7), page 747.]

A formal test distinguishing between real and complex quantum mechanics (to be later extended to the case of quaternions) can be performed as follows. Consider a beam of particles impinging on a scatterer. Let  $\psi_1$  represent the state of the scattered particles, namely,  $\psi_1$  is the difference between the actual state  $\psi$  and the state  $\psi_0$  that we would have if the scatterer were absent. Assume that  $\psi_0$  is normalized to unit flux. Now, set a detector at a large distance  $R$  from the scatterer, and let  $\chi/R$  represent the state of a unit flux of particles passing through that detector. Then the cross section for scattering into that detector is defined as

$$\sigma_1 = |\langle \chi, \psi_1 \rangle|^2,$$

where  $\langle \chi, \psi \rangle$  denotes the scalar product of the states  $\chi$  and  $\psi$ . If this scalar product is a complex number, we can write

$$\langle \chi, \psi_1 \rangle = a_1 \exp(i\phi_1),$$

where  $a_1$  is real, so that

$$\sigma_1 = a_1^2.$$

Similar formulas hold for quaternion quantum theory, with  $\exp(i\phi_1)$  replaced by a unimodular quaternion.

Consider now a different scatterer, with scattering amplitude

$$\langle \chi, \psi_2 \rangle = a_2 \exp(i\phi_2).$$

We have likewise

$$\sigma_2 = |\langle \chi, \psi_2 \rangle|^2.$$

Finally, if both scatterers are present, we have to a good approximation

$$\langle \chi, \psi_{12} \rangle = \langle \chi, \psi_1 \rangle + \langle \chi, \psi_2 \rangle.$$

This relation is valid if double scattering can be neglected. The total cross section thus is

$$\sigma_{12} = |a_1 \exp(i\phi_1) + a_2 \exp(i\phi_2)|^2 = \sigma_1 + \sigma_2 + 2\sqrt{\sigma_1\sigma_2} \cos(\phi_1 - \phi_2).$$

Note that  $\sigma_{12}$  is well defined provided that the relative position of the scatterers is held fixed (coherent scattering).

Define

$$\gamma = (\sigma_{12} - \sigma_1 - \sigma_2)/2\sqrt{\sigma_1\sigma_2}.$$

This expression involves only observable cross sections, and can therefore be actually measured for any pair of scatterers. The measurement of  $\gamma$  thus gives a simple criterion for distinguishing between real and complex quantum theories:

*If  $\gamma = \pm 1$ , real quantum theory is admissible. If  $|\gamma| < 1$ , we may have complex (or quaternionic) quantum theory. And if  $|\gamma| > 1$ , the superposition principle is violated.*

Note that the above formulas have been derived for pure states, and they may not be valid for mixtures, e.g., for an unpolarized beam, if the cross sections are spin or polarization dependent. For such a mixture, we can only measure averages:

$$\langle \sigma_{12} \rangle = \langle \sigma_1 \rangle + \langle \sigma_2 \rangle + 2\langle \sqrt{\sigma_1\sigma_2} \cos(\phi_1 - \phi_2) \rangle.$$

In that case, we can still define an averaged  $\langle \gamma \rangle$  by

$$\langle \gamma \rangle = (\langle \sigma_{12} \rangle - \langle \sigma_1 \rangle - \langle \sigma_2 \rangle)/2\sqrt{\langle \sigma_1 \rangle \langle \sigma_2 \rangle} = \langle \sqrt{\sigma_1\sigma_2} \cos(\phi_1 - \phi_2) \rangle / \sqrt{\langle \sigma_1 \rangle \langle \sigma_2 \rangle}.$$

However, this  $\langle \gamma \rangle$  is not the cosine of a phase difference and some of the formulas derived above are not valid. (They do remain valid if the cross sections are not affected by the spin or polarization variables).

We now consider a third scatterer and define, as previously,  $\sigma_3$ ,  $\sigma_{31}$ , and  $\sigma_{32}$ , and also

$$\alpha = (\sigma_{23} - \sigma_2 - \sigma_3)/2\sqrt{\sigma_2\sigma_3},$$

and

$$\beta = (\sigma_{31} - \sigma_3 - \sigma_1)/2\sqrt{\sigma_3\sigma_1}.$$

In complex quantum theory,  $\alpha$ ,  $\beta$ , and  $\gamma$  are the cosines of  $(\phi_2 - \phi_3)$ ,  $(\phi_3 - \phi_1)$ , and  $(\phi_1 - \phi_2)$ , respectively, and therefore they are not independent, since these angles sum up to zero. An elementary calculation gives

$$F(\alpha, \beta, \gamma) := \alpha^2 + \beta^2 + \gamma^2 - 2\alpha\beta\gamma = 1.$$

On the other hand, if the amplitudes  $\langle \chi, \psi_n \rangle$  are quaternions, rather than ordinary complex numbers, they do not behave as vectors in a plane, but as vectors in a four-dimensional space. We then have  $0 \leq F(\alpha, \beta, \gamma) \leq 1$ . The criterion for distinguishing between complex and quaternionic quantum theory can thus be stated as follows:

*If  $F(\alpha, \beta, \gamma) = 1$ , complex quantum theory is admissible. If  $F < 1$ , we may have quaternion quantum theory. And if  $F > 1$ , the superposition principle is violated.*

Note that  $F$  can never be negative if  $\alpha^2 + \beta^2 + \gamma^2 \leq 3$ . It is interesting that no new information can be obtained by considering the three scatterers simultaneously, because  $\sigma_{123} = \sigma_{12} + \sigma_{23} + \sigma_{31} - \sigma_1 - \sigma_2 - \sigma_3$  for all types of quantum theory.

It is thus clear that quaternion quantum theory is essentially different from complex quantum theory. It is not equivalent to having a hidden “internal” degree of freedom (Moravcsik, 1986). Let us examine some experiments that could distinguish between complex and quaternionic quantum theories (Peres, 1979).

As explained above, the scatterers must act coherently and multiple scattering should be negligible. This rules out some tantalizing ideas, like scattering neutrinos from the three different quarks in baryons. Conceptually, the simplest test is Bragg scattering by crystals made of three different kinds of atoms. Indeed this test was performed long ago with X-rays: the fact that phase angles are coplanar is the basis of the multiple isomorphous replacement method, used to resolve the structure of proteins (Blundell and Johnson, 1976). However, X-ray diffraction involves only the interaction of photons and electrons, and we should not expect to observe there significant deviations from standard quantum theory.

On the other hand, nuclear forces are not as well understood as quantum electrodynamics, and several nontrivial tests can be devised. A simple one is to examine low energy proton-proton scattering: the Coulomb amplitude is exactly known, and the nuclear interaction, having short range, can be analyzed in terms of a few

phase shifts (only the  $S$ -wave is significant, for low enough energy). If quaternions are implicated in this process and the  $S$ -wave phase shift does not involve the same imaginary unit  $i$  as the  $i$  that appears in the Coulomb amplitude, it will be impossible to fit experimental cross sections by the standard quantum mechanical formulas. Indeed, it was found long ago that there were “distressing irregularities” in all the data below 10 MeV (Sher, Signell and Heller, 1970). It is naturally tempting to try to explain these discrepancies by fitting the data with quaternionic amplitudes. Unfortunately, this makes the discrepancy even larger! The correct explanation of this riddle turned out to be a combination of various relativistic and radiative effects, and corrections for the finite size of the proton (Peres, 1978).

More promising experiments are those involving only nuclear forces. For example, a nontrivial test could be neutron diffraction by crystals made of three different isotopes. Unfortunately, the latter must have large capture cross sections to give any appreciable “quaternionic” effect. Indeed, the neutron scattering amplitude can be written (Gasiorowicz, 1974) as

$$f = [\eta \sin 2\delta + i(1 - \eta \cos 2\delta)]/2k,$$

where  $k$  is the wave number of the neutron,  $\delta$  is the  $S$ -wave phase shift, and  $\eta$  is the elasticity parameter. Both  $\delta$  and  $(1 - \eta)$  are very small, as can be seen from the formulas for the scattering and absorption cross sections:

$$\sigma_s = \frac{4\pi}{k^2} \left[ \eta \sin^2 \delta + \left( \frac{1 - \eta}{2} \right)^2 \right],$$

and

$$\sigma_a = \pi(1 - \eta^2)/k^2.$$

For thermal neutrons,  $4\pi/k^2 \simeq 10^8$  b, so that  $\delta \simeq 10^{-4}$ .

We thus have approximately  $f = [\delta + i(1 - \eta)/2]/k$ , the phase of which will be nontrivial provided that  $(1 - \eta)$  has at least the same order of magnitude as  $\delta$ . This implies that  $\sigma_a$  should be of the order of  $10^4$  b or more, for at least two of the scatterers. Most materials have much smaller absorption cross sections, and their  $f$  is almost real. As a consequence, we always have  $F(\alpha, \beta, \gamma) \simeq 1$  and this experiment cannot distinguish between complex and quaternionic quantum theories.

Instead of Bragg scattering, another possibility is neutron interferometry. The latter involves only the forward-scattering amplitude, so that this test has less

generality, but it is easier to perform. Consider a plane wave  $\exp(i\mathbf{k} \cdot \mathbf{r})$ . Passage through a plate of thickness  $L$ , having  $n$  scatterers per unit volume, changes the amplitude into  $T \exp(i\Delta) \exp(i\mathbf{k} \cdot \mathbf{r})$ . The transmission coefficient  $T$  is due to reflections at the surfaces of the plate. The macroscopic phase shift  $\Delta$  is given by

$$\Delta = [\eta \sin 2\delta + i(1 - \eta \cos 2\delta)] \pi n L / k^2.$$

The real part of  $\Delta$  is due to the change in optical path, and the imaginary part to absorption in the plate. By means of interference with a reference beam  $\exp(i\mathbf{k}' \cdot \mathbf{r})$ , with  $\mathbf{k}' \simeq \mathbf{k}$ , it is possible to measure both effects.

Now consider two plates made of different materials, taken singly and jointly. The total transmission coefficient  $T_{12}$  will not, in general, be  $T_1 T_2$  because of multiple reflections between the plates. However, the total phase shift  $\Delta_{12}$  ought to be  $\Delta_1 + \Delta_2$ , if our use of complex numbers is legitimate. On the other hand, quaternion interference usually implies  $\Delta_{12} \neq \Delta_1 + \Delta_2$ , because quaternion rotations do not commute. We therefore expect that the interference pattern may be affected by exchanging the order of two consecutive slabs made of different materials.

An experimental test, with thick slabs of titanium and aluminum, showed no such effect (Kaiser, George and Werner, 1984). This is not surprising, because both metals have low neutron absorption cross sections, while a strong absorption is needed for seeing quaternionic effects, as explained above. (The use of highly absorbing materials would have required a much higher neutron flux.) In spite of its negative result, this experiment had a remarkable feature: the introduction of both slabs in the neutron path yielded a higher fringe visibility than when a single slab was present! The reason for this curious behavior is that the very large phase shifts that were involved ( $+9860^\circ$  for Ti, and  $-9980^\circ$  for Al) were an appreciable fraction of the coherence length of the neutrons. The reduced visibility of the fringes when only one slab was introduced was due to the partial lack of coherence of the two neutron paths. That coherence was restored by the introduction of the second slab, with a nearly opposite phase shift.

Still another possible test could be a comparison of  $K_S$  regeneration (Perkins, 1972) produced by three different materials, taken singly and pairwise. Here, the observed quantity is the square of the forward regeneration amplitude. For our purpose, it is similar to a cross section, and the expression  $F(\alpha, \beta, \gamma)$  can be defined exactly as before. For this test, it would be especially interesting to compare neutron-rich and proton-rich nuclei, since they contain different ratios of up and down quarks. (Yet, if quaternionic effects occur at the level of individual nucleons, or individual quarks, we would still have only two different types of scatterers, rather than three as required.) This experiment has not yet been attempted.

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